Demystifying flow formulations

José Luis Izquierdo Zaragoza
josel.izquierdo@upct.es
http://ait.upct.es/~jlizquierdo

Politecnico di Milano

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Outline

Introduction

- Capacitated flow assignment (FA) problems are devoted to find a valid routing given:
  - Physical topology (nodes and links)
  - Link capacities
  - Set of traffic demands
- There are a number of models to solve FA problems. Some of them are:
  - Flow Formulation (FF)
  - Route Formulation (RF)
  - Source Formulation
  - Destination-based Formulation
- All in all, FF and RF are the most widely employed models in the literature. Now, let's describe them

Introduction

Network model

- First, we should define the network model and its associated notation
- Physical topology is assumed to be a multi-digraph, where links are unidirectional and it is allowed to have several parallel links between each node pair. Self-links are not allowed
- Traffic demands are assumed to be unidirectional and unicast (from one node to one destination). Self-demands are not allowed, but we can have several demands for each node pair
- FA problems with this network model are referred as to multi-commodity flow problems

Introduction

Notation

Element	Parameter	Description
Nodes	N	Set of nodes $n \in N$
	$\delta^+(n), \delta^-(n)$	Set of outgoing and incoming links from/to node n
Links	E	Set of links $e \in E$
	$a(e), b(e)$	Origin and destination nodes of link e
	l_e	Length of link e (Km)
	u_e	Capacity of link e (Erlangs)
	$\mathbf{u}$	Vector form of u_e
	y_e	Traffic carried by link e (Erlangs)
	$\mathbf{y}$	Vector form of y_e
Demands	D	Set of demands $d \in D$
	$a(d), b(d)$	Ingress and egress nodes of demand d
	h_d	Offered traffic for demand d
	$\mathbf{h}$	Vector form of h_d
	r_d	Carried traffic for demand d
	$\mathbf{r}$	Vector form of r_d
Routing	P	Set of paths $p \in P$
	$P_d \subseteq P$	Subset of the paths in P that are associated to demand d
	$P_e \subseteq P$	Subset of the paths in P that traverse link e
	x_p	Traffic volume carried by path p
	$\mathbf{x}$	Vector form of x_p
	$a(p), b(p), l(p)$	Origin and destination nodes, and number of hops of path p
	$d(p)$	Demand corresponding to path p

Outline

Modeling multi-commodity flow problems

- Multi-commodity flow (MCF) problems can be defined as follows: given a physical topology $G(N, E)$, link capacities u_e, and a set of traffic demands D, find a valid routing that minimizes/maximizes a certain function of the decision variables (routing variables)
- In FF models, routing is defined using x_{de} variables representing the traffic from demand $d \in D$ traversing link $e \in E$
- In RF models, routing is defined using x_p variables representing the traffic traversing path $p \in P$. A set of candidate paths must be pre-computed (i.e. using a k-shortest path algorithm)

Important: Each demand has its own (and dedicated) set of paths, so the carried traffic for a demand $d \in D$ is given by $\sum_{p \in P_d} x_p$

Modeling multi-commodity flow problems
Flow formulation

- Optimization model:

$$
\begin{aligned}
&\min \quad f(x_{de}) \\
&\sum_{e \in \delta^+(n)} x_{de} - \sum_{e \in \delta^-(n)} x_{de} = \begin{cases} h_d, & \text{if } n = a(d) \\ -h_d, & \text{if } n = b(d) \\ 0, & \text{otherwise} \end{cases} && \forall n \in N, d \in D && (1) \\
&\sum_{d \in D} \sum_{p \in P_d \cap P_e} x_{de} \leq u_e && \forall e \in E && (2) \\
&x_{de} \geq 0 && \forall d \in D, e \in E && (3)
\end{aligned}
$$

  where eq. (1) are the flow conservation constraints, eq. (2) are the link capacity constraints, and eq. (3) are the non-negativity constraints
- Total number of variables: $D \cdot E$
- Total number of constraints: $N \cdot D + E + D \cdot E$
- Pros: Routing is unconstrained ☺
- Cons: Routing may be ambiguous and not valid ☹

Modeling multi-commodity flow problems
Route formulation

- Optimization model:

$$\min \quad f(x_p)$$

$$\sum_{p \in P_d} x_p = h_d \qquad \forall d \in D \qquad (4)$$

$$\sum_{d \in D} \sum_{p \in P_d \cap P_e} x_p \leq u_e \qquad \forall e \in E \qquad (5)$$

$$x_p \geq 0 \qquad \forall p \in P \qquad (6)$$

where eq. (4) enforces that all offered traffic is carried for each demand, eq. (5) are the link capacity constraints, and eq. (6) are the non-negativity constraints

- Total number of variables: P
- Total number of constraints: $D + E + P$
- Pros: Routing is unambiguous ☺
- Cons: The optimality depends on the number of candidate paths for each demand ☹. A column generation approach may help

Modeling multi-commodity flow problems I

Comparison between FF and RF

- At the end of the day we will need routes to know where to send the traffic
- In RF, routes (or paths) are defined *a priori*, so we do not need any post-processing. In contrast, in FF we need a post-processing mechanism to get traffic routes from routing variables (x_{de})
- The question is clear: can we actually obtain the routing from x_{de} variables?
- The answer may be difficult: given a routing in the form of x_p variables it is straightforward (and univocal) to convert it to x_{de} variables using the following relation:

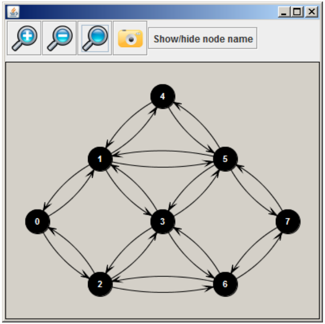
$$x_{de} = \sum_{p \in P_d \cap P_e} x_p \qquad \forall d \in D, e \in E \tag{7}$$

- Unfortunately, the reciprocal is not always true

Modeling multi-commodity flow problems II

Comparison between FF and RF

- In the following picture we show a reference network example. We assume each link has one unit of capacity, and there is only one traffic demand, from node 0 to node 7, requiring one unit of flow



Modeling multi-commodity flow problems III

Comparison between FF and RF

- Now, we solve the MCF (single-commodity in this case) problem using the following objective function:

$$\min \rho$$

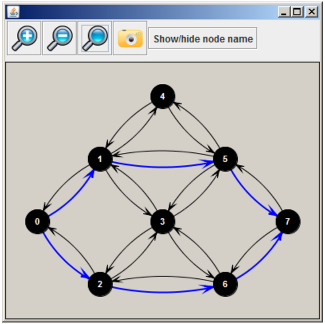
where $\rho \geq 0$ is the maximum link utilization, so we modify eq. (2) like this:

$$\sum_{d \in D} \sum_{p \in P_d \cap P_e} x_{de} \leq \rho \cdot u_e \qquad \forall e \in E \qquad (8)$$

Modeling multi-commodity flow problems IV

Comparison between FF and RF

- Clearly, there are two unambiguous routes for the traffic demand: (i) $0 \rightarrow 1 \rightarrow 5 \rightarrow 7$, and (ii) $0 \rightarrow 2 \rightarrow 6 \rightarrow 7$. In addition, both of them carrying $1/2$ units of flow



Modeling multi-commodity flow problems V

Comparison between FF and RF

- Now, let's try to solve the MCF (single-commodity in this case) problem using the following objective function:

$$\max \alpha$$

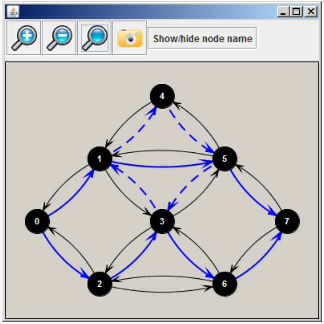
where $\alpha \geq 0$ is the scaling factor for the offered traffic, so that the source node sends as much traffic as possible while links are not becoming saturated. Here, we modify eq. (1) like this:

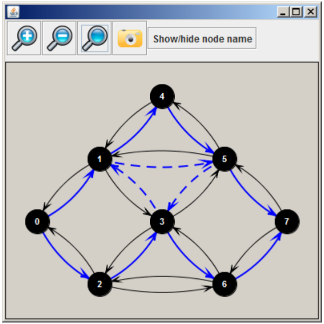
$$\sum_{e \in \delta^{+}(n)} x_{de} - \sum_{e \in \delta^{-}(n)} x_{de} = \begin{cases} \alpha \cdot h_d, & \text{if } n = a(d) \\ -\alpha \cdot h_d, & \text{if } n = b(d) \\ 0, & \text{otherwise} \end{cases} \qquad \forall n \in N, d \in D \qquad (9)$$

Modeling multi-commodity flow problems VI

Comparison between FF and RF

- In this case, a loop (or isolated cycle) appeared (and even it is not clear where). Please note that this solution is feasible since fulfills the constraints, but it has no sense from an engineering point of view





- There are two solutions: (i) add more constraints to the model (too complex), or (ii) use a post-processing algorithm

Outline

Getting a valid routing from x_{de} variables

Input: $G(N,E); D; x_{de}, \forall d \in D, e \in E$ **Output**: $P; d(p), x_p, \forall p \in P$

$P \leftarrow \emptyset$
$d(p) \leftarrow \emptyset$
$x_p \leftarrow \emptyset$
for each $d \in D$ **do**
    $P_d \leftarrow \emptyset$
    $E' \leftarrow \{e \in E | x_{de} > 0\}$
    while $E' \neq \emptyset$ **do**
        $p \leftarrow \textsc{ShortestPath}(G(N,E'), a(d), b(d))$
        if $p = \emptyset$ **then** break
        end if
        $P_d \leftarrow P_d \cup p$
        $\text{aux} = \min_{e \in p} x_{de}$
        $x_{de} = x_{de} - \text{aux} \quad \forall e \in p$
        $d(p) \leftarrow d(p) \cup d$
        $x_p \leftarrow x_p \cup \text{aux}$
    end while
    $P \leftarrow P \cup P_d$
end for
return $P, d(p), x_p$